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Quaternionic linear fractional transformations and direct isometries of $\mathbf{H}^5$

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Abstract

In the complex plane, an even number of reflection through lines or circles can be expressed in complex coordinates as a linear fractional transformation $w = (az+b)/(cz+d)$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. This also holds in $\mathbb{R}^4$: an even number of reflections through spheres or planes correspond to transformations $k = (ah + b)(ch + d)^{-1}$ with $a, b, c, d \in \mathbb{H}$. A theorem by Poincaré about direct isometries of hyperbolic spaces may therefore be rephrased: direct isometries of $\mathbf{H}^5$ correspond to quaternionic linear fractional transformations. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Möebius transformations sometime refer to linear fractional transformations with coefficients in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ [3], and sometime refer to the group of homeomorphisms of the one point compactification of $\mathbb{R}^n$ generated by reflections through spheres and planes. Here $\mathbf{M\ddot{o}b}_n$ will denote this last group, and $\mathbf{M\ddot{o}b}_n^+$ the subgroup of $\mathbf{M\ddot{o}b}_n$ consisting of transformations that preserve the orientation. Elements of $\mathbf{M\ddot{o}b}_n^+$ are those of $\mathbf{M\ddot{o}b}_n$ that may be written as a product of an even number of reflections.

It is well-known that an even number of reflections through lines and circles in the plane (i.e. elements of $\mathbf{M\ddot{o}b}_2^+$), when expressed in complex coordinates give linear fractional transformations $w = (az + b)/(cz + d)$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

In $\mathbb{R}^3$, let (x_1, x_2, x_3) denote the standard coordinates. The plane $x_3 = 0$ is identified with the complex plane. A reflection through a circle in the complex plane may be thought as the

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restriction of a reflection through a sphere whose center belongs to the plane $x_3 = 0$. In the upper-half space model for the Lobatchevski space $\mathbf{H}^3$, reflection through such spheres are isometries. This gives the connection between the group $\mathbf{M\ddot{o}b}_2$ and Isom ($\mathbf{H}^3$). Poincaré proved [5] that it holds in any dimension:

$$\mathbf{M\ddot{o}b}_n \sim \mathrm{Isom}(\mathbf{H}^{n+1}).$$

Direct isometries correspond to an even number of reflections.

The aim of this paper is to detail the correspondence between linear fractional transformation with coefficients in $\mathbb{H}$ and reflections through planes and spheres in $\mathbb{R}^4$.

By application of Poincaré's theorem, this will provide a description of direct isometries of $\mathbf{H}^5$ in terms of linear fractional maps with coefficients in $\mathbb{H}$.

2. $\mathbb{H}\mathrm{P}^1$

In the complex case, linear fractional maps correspond to the projection on $\mathbb{C}\mathrm{P}^1$ of linear mappings of $\mathbb{C}^2$ (see [4]). The same construction can be done in the quaternionic case although some care must be taken because of non-commutativity.

Definition 1. Define on $\mathbb{H}^2 \setminus \{(0,0)\}$ the following binary relation:

$$(h_1, h_2) \equiv (k_1, k_2) \Leftrightarrow \exists q \in \mathbb{H}^* \quad s.a. \quad h_1 = k_1 q, \quad h_2 = k_2 q.$$

We denote $\mathbb{H}\mathrm{P}^1$ the quotient space, and call it the (right) quaternionic projective space.

The real dimension of this manifold is 4. We use homogeneous coordinates $(h_1 : h_2)$, and a map from $\mathbb{H}\mathrm{P}^1 \backslash \{(1:0)\}$ to $\mathbb{H}$:

$$\begin{array}{lll} \pi & \mathbb{H}\mathrm{P}^1 \backslash \{(1:0)\} & \mapsto \mathbb{H} \\ & (h_1 : h_2) & \mapsto h_1 h_2^{-1}. \end{array}$$

Definition 2. A mapping of the form $\Phi(h_1, h_2) = (ah_1 + bh_2, ch_1 + dh_2)$ will be called a (left-)linear mapping (of $\mathbb{H}^2$). When $(0, 0)$ is the only solution of the system

$$au + bv = 0,$$
$$cu + dv = 0,$$

we say that Φ is non-degenerate.

Let now Φ be a non-degenerate linear mapping. One checks easily that Φ is a homeomorphism of $\mathbb{H}^2$. Its inverse does not look like one would expect. For instance, if a, b, c, d are non-zero quaternions, such that $\Phi(h_1, h_2) = (ah_1 + bh_2, ch_1 + dh_2)$ is non-degenerate, then

$$\Phi^{-1}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} (a - bd^{-1}c)^{-1}k_1 - (db^{-1}a - c)^{-1}k_2 \\ (b - ac^{-1}d)^{-1}k_1 - (ca^{-1}b - d)^{-1}k_2 \end{pmatrix}.$$

In this case, Φ induces a homeomorphism on $\mathbb{H}\mathrm{P}^1$. Such homeomorphisms can be seen through the projection π and produce linear fractional mappings of $\mathbb{H}$. As in the complex case, the pole of this mapping correspond to the point of $\mathbb{H}\mathrm{P}^1$ whose image is $(1:0)$.

Definition 3. A quaternionic (left-)linear fractional map is an application

$$\begin{aligned}\varphi \quad \mathbb{H} &\mapsto \mathbb{H},\\ h &\mapsto (ah+b)(ch+d)^{-1},\end{aligned}$$

with $a, b, c, d \in \mathbb{H}$ be such that the corresponding linear mapping $\Phi(h_1, h_2) = (ah_1 + bh_2, ch_1 + dh_2)$ is non-degenerate.

The set of these transformations form a group under composition.

3. The group **Möb$_4$**

Consider first R_S the reflection through the unit sphere S of $\mathbb{R}^4$. We denote (x_1, x_2, x_3, x_4) the standard coordinates in $\mathbb{R}^4$. Let x denote the quaternion with components (x_1, x_2, x_3, x_4) in the base $1, i, j, k$. The reflection R_S can be expressed as

$$R_S(x) = \frac{1}{\bar{x}},$$

where $\bar{x}$ denotes the quaternionic conjugation. If S denotes a sphere with center $c \in \mathbb{H}$ and radius $r > 0$ then the reflection through $S : R_S$ has a similar expression

$$R_S(x) = c + r^2(\bar{x} - \bar{c})^{-1}.$$

Let now R_P denote the reflection through a hyperplane P orthogonal to $a \neq 0$ and containing q. The quaternion a can be chosen such that $|a| = 1$.

$$R_P(x) = -a(\bar{x} - \bar{q})a + q.$$

One checks immediately that the composition of an even number of reflections through planes or spheres can be expressed as a map $\varphi(h) = (ah+b)(ch+d)^{-1}$ with $a, b, c, d \in \mathbb{H}$.

Let for instance R_1 and R_2 be two reflections through spheres with centers q_1 and q_2 and radii r_1 and r_2.

$$\begin{aligned}R_2 \circ R_1(x) &= c_2 + r_2^2\left(\bar{c}_1 + r_1^2(x - c_1)^{-1} - \bar{c}_2\right)^{-1}\\ &= c_2 + r_2^2(x - c_1)\left((\bar{c}_1 - \bar{c}_2)x + r_1^2 - |c_1|^2 + \bar{c}_2 c_1\right)^{-1}\\ &= \frac{\left(c_2(\bar{c}_1 - \bar{c}_2) + r_2^2\right)x + \left(r_1^2 c_2 - r_2^2 c_1 - c_2(\bar{c}_1 - \bar{c}_2)c_1\right)}{(\bar{c}_1 - \bar{c}_2)x + \left(r_1^2 - (\bar{c}_1 - \bar{c}_2)c_1\right)},\end{aligned}$$

with the convention that $a/b = ab^{-1}$.

In the plane, the composition of an even number of reflections through lines and circles give rise to maps $w = (az + b)/(cz + d)$ with $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

We saw that reflections through spheres and planes in $\mathbb{R}^4$ are expressed in quaternionic coordinates as maps $k = (ah + b)(ch + d)^{-1}$, with $a, b, c, d \in \mathbb{H}$. Here again a, b, c and d satisfy some extra condition. But because of non-commutativity, this extra condition is slightly different: it says that $(0, 0)$ is the only solution of

$$\begin{aligned} &au + bv = 0,\\ &cu + dv = 0. \end{aligned}$$

Such systems are called non-degenerate. Simple calculation show that a CNS for this system to be non-degenerated, is that

$$\begin{aligned} &c \neq 0,\\ &b \neq ac^{-1}d, \end{aligned} \quad \text{or} \quad \begin{aligned} &c = 0,\\ &ad \neq 0. \end{aligned}$$

Let us check that the coefficients of $R_2 \circ R_1$ match this condition. Let us assume first that $c_1 \neq c_2$, and let $\kappa = (\bar{c}_1 - \bar{c}_2)$. We have to check that

$$\begin{aligned} &\left(c_2\kappa + r_2^2\right) X + \left(r_1^2 c_2 - r_2^2 c_1 - c_2\kappa c_1\right) Y = 0,\\ &\kappa X + \left(r_1^2 - \kappa c_1\right) Y = 0, \end{aligned}$$

has only trivial solution. And since we have assumed that $\kappa \neq 0$, we have to just verify that

$$(c_2\kappa + r_2^2)k^{-1}(r_1^2 - \kappa c_1) \neq (r_1^2 c_2 - r_2^2 c_1 - c_2\kappa c_1),$$

but

$$(c_2\kappa + r_2^2)k^{-1}(r_1^2 - \kappa c_1) = (r_1^2 c_2 - r_2^2 c_1 - c_2\kappa c_1) + r_1^2 r_2^2 \kappa^{-1}.$$

Since r_1 and r_2 are strictly positive, $r_1^2 r_2^2 \kappa^{-1} \neq 0$, and the system has only trivial solution.

Suppose now that $c_1 = c_2$. Then

$$R_2 \circ R_1(x) = \frac{r_2^2}{r_1^2}(x - c_1) + c_2,$$

and the corresponding system has only trivial solution.

We thus have proved the following:

Proposition 1. *An even number of reflections through spheres and planes in $\mathbb{R}^4$ is a linear fractional transformation.*

In the sequel we prove that the converse is true: suppose that $\varphi(x) = (ax+b)(cx+d)^{-1}$ is a linear fractional transformation, φ can be decomposed into the product of an even number of reflections through planes or spheres in $\mathbb{R}^4$.

Lemma 1. *The following mappings belong to $\mathbf{M\ddot{o}b}_4^+$, which means that they may be decomposed into the product of an even number of reflections:*

1. $h \to h^{-1}$.
2. $h \to h + a$, *for any quaternion* a.
3. $h \to ah$ *and* $h \to ha$ *for any non-zero quaternion.*

Proof. Let τ_a be the translation of vector a in $\mathbb{R}^4$: it can be realised as the composition of two reflections through two parallel planes orthogonal to a and such that the distance between the two is equal to $|a|/2$.

Indeed, we saw that the reflection R_P through a plane P containing $q \in \mathbb{H}$ and orthogonal to a unit quaternion u may be expressed in quaternionic coordinates:

$$R_P(x) = -u(\bar{x} - \bar{q})u + q.$$

Let R_1 be the reflection through the plane containing O and orthogonal to a and R_2 be the reflection through the plane containing $q = a/2$ and orthogonal to a. Let $u = a/|a|$, we have

$$R_1(x) = -u\bar{x}u,$$
$$R_2(x) = -u\left(\bar{x} - \frac{\bar{a}}{2}\right)u + \left(\frac{a}{2}\right),$$

thus

$$R_2 \circ R_1(x) = x + \left(\frac{u\bar{a}}{2u}\right) + \left(\frac{a}{2}\right) = x + a.$$

The quaternionic inversion $h \to h^{-1}$ result from one reflection through the unit sphere, and quaternionic conjugation, which may in turn be decomposed into three reflections through planes $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$.

Let now a be a unit quaternion. The mapping $\alpha(h) = ah$ is orthogonal with determinant 1. Let A denote the matrix associated to α with respect to the standard basis of $\mathbb{H}$. The characteristic ploynomial of A is $p(t) = (1 - \mathrm{Re}(a)t + t^2)^2$, where $\mathrm{Re}(a)$ denotes the real part of a. Thus, there is an orthonormal basis relative to which the matrix of α is of the following form:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

This matrix decomposes into the product of four reflections. To see this, first remember that any rotation in $\mathbb{R}^2$ decomposes into the product of two reflections. Here we have

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} = R_1 R_2,$$

with

$$R_1 = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

It is thus enough to show that R_1 decomposes into two reflections. But this follows directly from what we have said about rotations in the plane. Let $u = (-\sin\varphi/2,\ \cos\varphi/2, 0, 0)$ and $v = (0, 1, 0, 0)$. Simple calculation shows that

$$R_u \circ R_v = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let now a be a strictly positive real number. In the complex plane, the mapping $z \to az$ corresponds to two reflections through concentric circles. In $\mathbb{R}^4$ the transformation $h \to ah$ corresponds to the composition of two reflections through concentric spheres. Indeed, if we choose r_1 and r_2 such that $r_2^2/r_1^2 = a$, and if S_1 and S_2 denote reflections through concentric spheres centered at the origin with radius r_1 and r_2, then,

$$S_2 \circ S_1(h) = \frac{r_2^2}{r_1^2} h = ah.$$

If a is a strictly negative number, the mapping $h \to ah$ decomposes into four reflections through the coordinate planes and two reflections through concentric spheres centered at the origin.

Let now a be any non-zero quaternion, writing $a = |a|u$, where u is a unit quaternion, one gets the requested decomposition for the mapping $h \to ah$.

The mappings $h \to ha$ can be decomposed in the same fashion. □

The previous demonstration is more or less an adaptation of the decomposition of similitudes in the complex plane into product of reflections through circles and lines.

Theorem 1. *Any linear fractional mapping can be decomposed into the product of an even number of reflections through planes and spheres of* $\mathbb{R}^4$.

Proof. Let φ be a linear fractional mapping.

$$\varphi(x) = (ah + b)(ch + d)^{-1} \quad \text{with } a, b, c, d \in \mathbb{H}.$$

Assume first that $c \neq 0$, then $b - ac^{-1}d$ cannot be zero because if it was, the system

$$au + bv = 0,$$
$$cu + dv = 0,$$

would have a non-trivial solution: $(-c^{-1}d, 1)$. The following transformations may thus be decomposed into an even number of reflections, and their composition gives φ:

$$h \to ch,$$
$$h \to h + d,$$
$$h \to h^{-1},$$
$$h \to (b - ac^{-1}d)h,$$
$$h \to h + ac^{-1}.$$

Let us assume now that $c = 0$. The coefficients a and d cannot be 0 because φ is not degenerate. Thus

$$\varphi(h) = ahd^{-1} + bd^{-1},$$

which can be decomposed into an even number of reflections. □

We finally have, as announced, the following result.

Theorem 2. *The set of quaternionic linear fractional transformations form a group under composition which is isomorphic to* $\mathbf{M\ddot{o}b}_4^+$, *and to the group of direct isometries of* $\mathbf{H}^5$.

This result is meant to be added to the list of trinities that V.I. Arnold presented in [1] and [2].

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